

GENERALIZATIONS OF GUESSAB–SCHMEISSER FORMULA VIA FINK TYPE IDENTITY WITH APPLICATIONS TO QUADRATURE RULES

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ABSTRACT. In this work, an expansion of Guessab–Schmeisser two points formula for n -times differentiable functions via Fink type identity is established. Generalization of the main result for harmonic sequence of polynomials is established. Several bounds of the presented results are proved. As applications, some quadrature rules are elaborated and discussed. Error bounds of the presented quadrature rules via Chebyshev–Grüss type inequalities are also provided.

1. INTRODUCTION

For a continuous function f defined on $[a, b]$, the integral mean-value theorem (IMVT) guarantees an $x \in [a, b]$ such that

$$(1.1) \quad f(x) = \frac{1}{b-a} \int_a^b f(t) dt.$$

In order to measure the difference between any value of f in $[a, b]$ and its weighted value, Ostrowski in his celebrated work [45] established a very interesting inequality for differentiable functions with bounded derivatives which in connection with (1.1), which reads:

Theorem 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , the interior of the interval I , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $\|f'\|_\infty = \sup_{x \in [a, b]} |f'(x)| \leq \infty$. Then, the inequality*

$$(1.2) \quad \left| (b-a)f(x) - \int_a^b f(u) du \right| \leq \left[\frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty,$$

holds for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller ones.

In 1976 Milovanović and Pečarić [43] presented their famous generalization of (1.1) via Taylor series, where they proved that:

$$(1.3) \quad \left| \frac{1}{n} \left(f(x) + \sum_{k=1}^{n-1} F_k(x) \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq C(n, \infty, x) \|f^{(n)}\|_\infty,$$

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such that

$$(1.4) \quad F_k(x) = \frac{n-k}{n!} \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}.$$

In fact, Milovanović and Pečarić proved the case that

$$C(n, \infty, x) = \frac{(x-a)^{n+1} + (b-x)^{n+1}}{(b-a)n(n+1)!}.$$

In 1992, Fink studied (1.3) in different point of view, he introduced a new representation of real n -times differentiable function whose n -th derivative ($n \geq 1$) is absolutely continuous by combining Taylor series and Peano kernel approach together. Namely, in [29] we find:

$$(1.5) \quad \frac{1}{n} \left(f(x) + \sum_{k=1}^{n-1} F_k \right) - \frac{1}{b-a} \int_a^b f(y) dy \\ = \frac{1}{n!(b-a)} \int_a^b (x-t)^{n-1} p(t, x) f^{(n)}(t) dt,$$

for all $x \in [a, b]$, where

$$(1.6) \quad p(t, x) = \begin{cases} t-a, & t \in [a, x] \\ t-b, & t \in [x, b] \end{cases}.$$

In the same work, Fink proved the following bound of (1.5).

$$(1.7) \quad \left| \frac{1}{n} \left(f(x) + \sum_{k=1}^{n-1} F_k(x) \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq C(n, p, x) \|f^{(n)}\|_p$$

where $\|\cdot\|_r$, $1 \leq r \leq \infty$ are the usual Lebesgue norms on $L_r[a, b]$, i.e.,

$$\|f\|_\infty := \operatorname{ess\,sup}_{t \in [a, b]} |f(t)|,$$

and

$$\|f\|_r := \left(\int_a^b |f(t)|^r dt \right)^{1/r}, \quad 1 \leq r < \infty,$$

such that

$$C(n, p, x) = \frac{\left[(x-a)^{nq+1} + (b-x)^{nq+1} \right]^{1/q}}{(b-a)n!} B^{1/q}((n-1)q+1, q+1),$$

for $1 < p \leq \infty$, $B(\cdot, \cdot)$ is the beta function, and for $p = 1$

$$C(n, 1, x) = \frac{(n-1)^{n-1}}{(b-a)n^n n!} \max \{ (x-a)^n, (b-x)^n \}.$$

All previous bounds are sharp.

Indeed Fink representation can be considered as the first elegant work (after Darboux work [39], p.49) that combines two different approaches together, so that Fink representation is not less important than Taylor expansion itself. So that, many authors were interested to study Fink representation approach, more detailed and related results can be found in [1],[2],[13],[14] and [20].

In 2002 and the subsequent years after that, the Ostrowski's inequality entered in a new phase of modifications and developments. A new inequality of Ostrowski's type was born, where Guessab and Schmeisser in [36] discussed an inequality from algebraic and analytic points of view which is in connection with Ostrowski inequality; called '*the companion of Ostrowski's inequality*' as suggested later by Dragomir in [26]. The main part of Guessab–Schmeisser inequality reads the difference between symmetric values of a real function f defined on $[a, b]$ and its weighed value, i.e.,

$$\frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt, \quad x \in \left[a, \frac{a+b}{2}\right].$$

Namely, in the significant work [36] we find the first primary result is that:

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be satisfies the Hölder condition of order $r \in (0, 1]$. Then for each $x \in [a, \frac{a+b}{2}]$, the we have the inequality*

$$(1.8) \quad \left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{M}{b-a} \frac{(2x-2a)^{r+1} + (a+b-2x)^{r+1}}{2^r(r+1)}.$$

This inequality is sharp for each admissible x . Equality is attained if and only if $f = \pm M f_ + c$ with $c \in \mathbb{R}$ and*

$$f_*(t) = \begin{cases} (x-t)^r, & \text{if } a \leq t \leq x \\ (t-x)^r, & \text{if } x \leq t \leq \frac{a+b}{2} \\ f_*(a+b-t), & \text{if } \frac{a+b}{2} \leq t \leq b \end{cases}.$$

In the same work [36], the authors discussed and investigated (1.8) for other type of assumptions. Among others, a brilliant representation (or identity) of n -times differentiable functions whose n -th derivatives are piecewise continuous was established as follows:

Theorem 3. *Let f be a function defined on $[a, b]$ and having there a piecewise continuous n -th derivative. Let Q_n be any monic polynomial of degree n such that $Q_n(t) = (-1)^n Q_n(a+b-t)$. Define*

$$K_n(t) = \begin{cases} (t-a)^n, & \text{if } a \leq t \leq x \\ Q_n(t), & \text{if } x \leq t \leq a+b-x \\ (t-b)^n, & \text{if } a+b-x \leq t \leq b \end{cases}.$$

Then,

$$(1.9) \quad \int_a^b f(t) dt = (b-a) \frac{f(x) + f(a+b-x)}{2} + E(f; x)$$

where,

$$E(f; x) = \sum_{\nu=1}^{n-1} \left[\frac{(x-a)^{\nu+1}}{(\nu+1)!} - \frac{Q_n^{(n-\nu-1)}(x)}{n!} \right] \left[f^{(\nu)}(a+b-x) + (-1)^\nu f(x) \right] + \frac{(-1)}{n!} \int_a^b K_n(t) f^{(n)}(t) dt.$$

This generalization (1.9) can be considered as a companion type expansion of Euler–Maclaurin formula that expand symmetric values of real functions. In this way, families of various quadrature rules can be presented, as shown -for example- in [38]. Therefore, since 2002 and after the presentation of (1.8), several authors have studied, developed and established new presentations concerning (1.8) using several approaches and different tools, for this purpose see the recent survey [25].

Far away from this, in the last thirty years the concept of harmonic sequence of polynomials or Appell polynomials have been used at large in numerical integrations and expansions theory of real functions. Let us recall that, a sequence of polynomials $\{P_k(t, \cdot)\}_{k=0}^\infty$ satisfies the Appell condition (see [12]) if $\frac{\partial}{\partial t} P_k(t, \cdot) = P_{k-1}(t, \cdot)$ ($\forall k \geq 1$) with $P_0(t, \cdot) = 1$, for all well-defined order pair (t, \cdot) . A slightly different definition was considered in [42].

In 2003, motivated by work of Matić et. al. [42], Dedić et. al. in [20], introduced the following smart generalization of Ostrowski's inequality via harmonic sequence of polynomials:

$$(1.10) \quad \frac{1}{n} \left[f(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1} \widetilde{F}_k(a, b) \right] = \frac{(-1)^{n-1}}{(b-a)n} \int_a^b P_{n-1}(t) p(t, x) f^{(n)}(t) dt,$$

where P_k is a harmonic sequence of polynomials satisfies that $P'_k = P_{k-1}$ with $P_0 = 1$,

$$(1.11) \quad \widetilde{F}_k(a, b) = \frac{(-1)^k (n-k)}{b-a} \left[P_k(a) f^{(k-1)}(a) - P_k(b) f^{(k-1)}(b) \right]$$

and $p(t, x)$ is given in (1.5). In particular, if we take $P_k(t) = \frac{(t-x)^k}{k!}$ then we refer to Fink representation (1.5).

In 2005, Dragomir [26] proved the following bounds of the companion of Ostrowski's inequality for absolutely continuous functions.

Theorem 4. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then we have the inequalities*

$$(1.12) \quad \left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \begin{cases} \left[\frac{1}{8} + 2 \left(\frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty, & f' \in L_\infty[a, b] \\ \frac{2^{1/q}}{(q+1)^{1/q}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} - \left(\frac{\frac{a+b}{2} - x}{b-a} \right)^{q+1} \right]^{1/q} (b-a)^{1/q} \|f'\|_{[a,b],p}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \text{ and } f' \in L_p[a, b] \\ \left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right] \|f'\|_{[a,b],1} \end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$. The constants $\frac{1}{8}$ and $\frac{1}{4}$ are the best possible in (1.12) in the sense that it cannot be replaced by smaller constants.

The author of this paper have took a serious attention to Guessab–Schmeisser inequality in the works [3]–[11]. For other related results and generalizations concerning Ostrowski’s inequality and its applications we refer the reader to [13]–[18], [24], [26]–[28], [37], [40], [47] and [48].

In the last fifteen years, constructions of quadrature rules using expansion of an arbitrary function in Bernoulli polynomials and Euler–Maclaurin’s type formulae have been established, improved and investigated. These approaches permit many researchers to work effectively in the area of numerical integration where several error approximations of various quadrature rules presented with high degree of exactness. Mainly, works of Dedić et al. [20]–[24], Aljinović et al. [1], [2], Kovać et al. [38] and others, received positive responses and good interactions from other focused researchers. Among others, Franjić et al. in several works (such as [30]–[34]) constructed several Newton–Cotes and Gauss quadrature type rules using a certain expansion of real functions in Bernoulli polynomials or Euler–Maclaurin’s type formulae.

Unfortunately, the expansions (1.5), (1.9) and (1.10) have not been used to construct quadrature rules yet. It seems these expansions were abandoned or neglected in literature because most of authors are still use the classical Euler–Maclaurin’s formula and expansions in Bernoulli polynomials.

This work has several aims and goals, the first aim is to generalize Guessab–Schmeisser two points formula for n -times differentiable functions via Fink type identity and provide several type of bounds for the remainder formula. The second goal, is to highlight the importance of these expansions and give a serious attention to their applicable usefulness in constructing various quadrature rules. The third aim, is to spotlight the role of Čebyšev functional in integral approximations.

This work is organized as follows: in the next section, a Guessab–Schmeisser two points formula for n -times differentiable functions via Fink type identity is established. Bounds for the remainder term of the presented formula are proved. In section 3, bounds for the remainder term via Chebyshev–Grüss type inequalities are presented. In section 4, generalizations of the obtained results to harmonic

sequence of polynomials are given. In section 5, representations of some quadrature rules are introduced and their errors are explored.

2. THE RESULTS

2.1. Guessab–Schmeisser formula via Fink type identity.

Theorem 5. *Let I be a real interval, $a, b \in I^\circ$ ($a < b$). Let $f : I \rightarrow \mathbb{R}$ be n -times differentiable on I° such that $f^{(n)}$ is absolutely continuous on I° with $(\cdot - t)^{n-1} S(t, \cdot) f^{(n)}(t)$ is integrable. Then we have the representation*

$$(2.1) \quad \frac{1}{n} \left(\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} G_k \right) - \frac{1}{b-a} \int_a^b f(y) dy \\ = \frac{1}{n!(b-a)} \int_a^b (x-t)^{n-1} S(t, x) f^{(n)}(t) dt,$$

for all $x \in [a, \frac{a+b}{2}]$, where

$$(2.2) \quad G_k := G_k(x) = \frac{(n-k)}{k!(b-a)} \cdot \left\{ (x-a)^k \left[f^{(k-1)}(a) + (-1)^{k+1} f^{(k-1)}(b) \right] \right. \\ \left. + \left(1 + (-1)^{k+1} \right) \left(\frac{a+b}{2} - x \right)^k f^{(k-1)} \left(\frac{a+b}{2} \right) \right\},$$

and

$$(2.3) \quad S(t, x) = \begin{cases} t-a, & t \in [a, x] \\ t - \frac{a+b}{2}, & t \in (x, a+b-x) \\ t-b, & t \in [a+b-x, b] \end{cases}.$$

Proof. Fix $x \in [a, b]$. Starting with Taylor series expansion for f along $[a, \frac{a+b}{2}]$

$$(2.4) \quad f(x) = f(y) + \sum_{k=1}^{n-1} \frac{f^{(k)}(y)}{k!} (x-y)^k + \frac{1}{(n-1)!} \int_y^x (x-t)^{n-1} f^{(n)}(t) dt.$$

Integrating with respect to y along $[a, \frac{a+b}{2}]$, we have

$$(2.5) \quad \frac{b-a}{2} f(x) = \int_a^{\frac{a+b}{2}} f(y) dy + \sum_{k=1}^{n-1} \frac{1}{k!} \int_a^{\frac{a+b}{2}} (x-y)^k f^{(k)}(y) dy \\ + \frac{1}{(n-1)!} \int_a^{\frac{a+b}{2}} \left(\int_y^x (x-t)^{n-1} f^{(n)}(t) dt \right) dy.$$

Also, for $x \in [\frac{a+b}{2}, b]$, f has the representation

$$(2.6) \quad f(a+b-x) = f(y) + \sum_{k=1}^{n-1} \frac{f^{(k)}(y)}{k!} (a+b-x-y)^k \\ + \frac{1}{(n-1)!} \int_y^{a+b-x} (a+b-x-t)^{n-1} f^{(n)}(t) dt.$$

Integrating with respect to y along $[\frac{a+b}{2}, b]$, we have

$$(2.7) \quad \frac{b-a}{2} f(a+b-x) = \int_{\frac{a+b}{2}}^b f(y) dy + \sum_{k=1}^{n-1} \frac{1}{k!} \int_{\frac{a+b}{2}}^b (a+b-x-y)^k f^{(k)}(y) dy \\ + \frac{1}{(n-1)!} \int_{\frac{a+b}{2}}^b \left(\int_y^{a+b-x} (a+b-x-t)^{n-1} f^{(n)}(t) dt \right) dy$$

Adding (2.5) and (2.7), we get

$$(2.8) \quad (b-a) \frac{f(x) + f(a+b-x)}{2} \\ = \int_a^b f(y) dy + \sum_{k=1}^{n-1} I_k \\ + \frac{1}{(n-1)!} \left[\int_a^{\frac{a+b}{2}} \left(\int_y^x (x-t)^{n-1} f^{(n)}(t) dt \right) dy \right. \\ \left. + \int_{\frac{a+b}{2}}^b \left(\int_y^{a+b-x} (a+b-x-t)^{n-1} f^{(n)}(t) dt \right) dy \right],$$

where, $I_k = J_k + h_k$, $I_0 = \int_a^b f(y) dy$, $J_k = \frac{1}{k!} \int_a^{\frac{a+b}{2}} (x-y)^k f^{(k)}(y) dy$ and $h_k = \frac{1}{k!} \int_{\frac{a+b}{2}}^b (a+b-x-y)^k f^{(k)}(y) dy$ ($k \geq 1$). Therefore, the following recurrence relations follows using integration by parts formula (see [43]):

$$(2.9) \quad (n-k)(J_k - J_{k-1}) = -(b-a)D_k, \quad (1 \leq k \leq n-1),$$

where,

$$D_k = \frac{(n-k)}{k!} \cdot \frac{(x-a)^k f^{(k-1)}(a) - (x - \frac{a+b}{2})^k f^{(k-1)}(\frac{a+b}{2})}{b-a}.$$

Similarly we have

$$(2.10) \quad (n-k)(\ell_k - \ell_{k-1}) = -(b-a)L_k, \quad (1 \leq k \leq n-1)$$

where,

$$L_k = \frac{(n-k)}{k!} \cdot \frac{(\frac{a+b}{2} - x)^k f^{(k-1)}(\frac{a+b}{2}) - (a-x)^k f^{(k-1)}(b)}{b-a}.$$

Therefore, by adding (2.9) and (2.10) we get

$$(2.11) \quad (n-k)(I_k - I_{k-1}) = -(b-a)G_k, \quad (1 \leq k \leq n-1)$$

where $G_k = D_k + L_k$.

Summing the terms in (2.11) from $k=1$ up to $k=n-1$, simplifications lead us to write

$$(2.12) \quad \sum_{k=1}^{n-1} I_k = -(b-a) \sum_{k=1}^{n-1} G_k + (n-1)I_0.$$

Substituting (2.12) in (2.8) and rearrange the terms we get that

$$\begin{aligned}
 (2.13) \quad & \frac{1}{n} \left(\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} G_k \right) - \frac{1}{b-a} \int_a^b f(y) dy \\
 &= \frac{1}{n!(b-a)} \left[\int_a^{\frac{a+b}{2}} \left(\int_y^x (x-t)^{n-1} f^{(n)}(t) dt \right) dy \right. \\
 &\quad \left. + \int_{\frac{a+b}{2}}^b \left(\int_y^{a+b-x} (a+b-x-t)^{n-1} f^{(n)}(t) dt \right) dy \right].
 \end{aligned}$$

To simplify the right hand side, we write

$$\begin{aligned}
 (2.14) \quad & \int_a^{\frac{a+b}{2}} dy \int_y^x dt = \int_a^x dy \int_y^x dt + \int_x^{\frac{a+b}{2}} dy \int_y^x dt \\
 &= \int_a^x dt \int_a^t dy - \int_x^{\frac{a+b}{2}} dy \int_y^x dt \\
 &= \int_a^x dt \int_a^t dy - \int_x^{\frac{a+b}{2}} dt \int_t^{\frac{a+b}{2}} dy,
 \end{aligned}$$

and

$$\begin{aligned}
 (2.15) \quad & \int_{\frac{a+b}{2}}^b dy \int_y^{a+b-x} dt = \int_{\frac{a+b}{2}}^{a+b-x} dy \int_y^{a+b-x} dt + \int_{a+b-x}^b dy \int_y^{a+b-x} dt \\
 &= \int_{\frac{a+b}{2}}^{a+b-x} dt \int_{\frac{a+b}{2}}^t dy - \int_{\frac{a+b}{2}}^b dy \int_y^{a+b-x} dt \\
 &= \int_{\frac{a+b}{2}}^{a+b-x} dt \int_{\frac{a+b}{2}}^t dy - \int_{a+b-x}^b dt \int_t^b dy.
 \end{aligned}$$

Adding (2.14) and (2.15), we get

$$\begin{aligned}
 (2.16) \quad & \int_a^{\frac{a+b}{2}} dy \int_y^x dt + \int_{\frac{a+b}{2}}^b dy \int_y^{a+b-x} dt \\
 &= \int_a^x dt \int_a^t dy - \int_x^{\frac{a+b}{2}} dt \int_t^{\frac{a+b}{2}} dy + \int_{\frac{a+b}{2}}^{a+b-x} dt \int_{\frac{a+b}{2}}^t dy - \int_{a+b-x}^b dt \int_t^b dy.
 \end{aligned}$$

In viewing (2.16), the right hand side of (2.13) becomes

$$\begin{aligned}
 & \frac{1}{n!(b-a)} \left[\int_a^{\frac{a+b}{2}} \left(\int_y^x (x-t)^{n-1} f^{(n)}(t) dt \right) dy \right. \\
 & \quad \left. + \int_{\frac{a+b}{2}}^b \left(\int_y^{a+b-x} (a+b-x-t)^{n-1} f^{(n)}(t) dt \right) dy \right] \\
 &= \frac{1}{n!(b-a)} \int_a^b (x-t)^{n-1} S(t, x) f^{(n)}(t) dt,
 \end{aligned}$$

where

$$S(t, x) = \begin{cases} t - a, & t \in [a, x] \\ t - \frac{a+b}{2}, & t \in (x, a+b-x) \\ t - b, & t \in [a+b-x, b] \end{cases}.$$

Thus, the identity (2.13) becomes

$$(2.17) \quad \frac{1}{n} \left(\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} G_k(x) \right) - \frac{1}{b-a} \int_a^b f(y) dy \\ = \frac{1}{n!(b-a)} \int_a^b (x-t)^{n-1} S(t, x) f^{(n)}(t) dt$$

for all $x \in [a, \frac{a+b}{2}]$. \square

Theorem 6. *Under the assumptions of Theorem 5. We have*

$$(2.18) \quad \left| \frac{1}{n} \left(\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} G_k(x) \right) - \frac{1}{b-a} \int_a^b f(y) dy \right| \\ \leq C(n, p, x) \|f^{(n)}\|_p$$

holds, where

$$(2.19) \quad M(n, p, x) = \begin{cases} \frac{1}{n \cdot n!(b-a)} \left(\frac{n-1}{n} \right)^{n-1} \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^n, & \text{if } p = 1 \\ \frac{2^{1/q}}{n!(b-a)} \left[(x-a)^{nq+1} + \left(\frac{a+b}{2} - x \right)^{nq+1} \right]^{1/q} \\ \quad \times B^{\frac{1}{q}}((n-1)q+1, q+1), & \text{if } 1 < p \leq \infty, q = \frac{p}{p-1} \end{cases}.$$

The constant $C(n, p, x)$ is the best possible in the sense that it cannot be replaced by a smaller ones.

Proof. Utilizing the triangle integral inequality on the identity (2.1) and employing some known norm inequalities we get

$$\left| \frac{1}{n} \left(\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} G_k(x) \right) - \frac{1}{b-a} \int_a^b f(y) dy \right| \\ \leq \frac{1}{n!(b-a)} \int_a^b |x-t|^{n-1} |S(t, x)| |f^{(n)}(t)| dt \\ \leq \begin{cases} \|f^{(n)}\|_1 \sup_{a \leq t \leq b} \left\{ |x-t|^{n-1} |k(t, x)| \right\}, & p = 1 \\ \|f^{(n)}\|_p \left(\int_a^b |x-t|^{(n-1)q} |k(t, x)|^q dt \right)^{1/q}, & 1 < p < \infty \\ \|f^{(n)}\|_\infty \int_a^b |x-t|^{n-1} |k(t, x)| dt, & p = \infty \end{cases}.$$

It is easy to find that for $p = 1$, we have

$$\begin{aligned} \sup_{a \leq t \leq b} \left\{ |x - t|^{n-1} |S(t, x)| \right\} &= \frac{1}{n} \left(\frac{n-1}{n} \right)^{n-1} \max \left\{ (x-a)^n, \left(\frac{a+b}{2} - x \right)^n \right\} \\ &= \frac{1}{n} \left(\frac{n-1}{n} \right)^{n-1} \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^n, \end{aligned}$$

and for $1 < p < \infty$, we have

$$\begin{aligned} &\int_a^b |x - t|^{(n-1)q} |S(t, x)|^q dt \\ &= \int_a^x |x - t|^{(n-1)q} (t-a)^q dt + \int_x^{a+b-x} |x - t|^{(n-1)q} \left| t - \frac{a+b}{2} \right|^q dt \\ &\quad + \int_{a+b-x}^b |x - t|^{(n-1)q} (b-t)^q dt \\ &= 2 \left[(x-a)^{nq+1} + \left(\frac{a+b}{2} - x \right)^{nq+1} \right] \left(\int_0^1 (1-s)^{(n-1)q} s^q ds \right) \\ &= 2 \left[(x-a)^{nq+1} + \left(\frac{a+b}{2} - x \right)^{nq+1} \right] B((n-1)q+1, q+1) \end{aligned}$$

where, we use the substitutions $t = (1-s)a + sx$, $t = (1-s)x + s(a+b-x)$ and $t = (1-s)(a+b-x) + sb$; respectively. The third case, $p = \infty$ holds by setting $p = \infty$ and $q = 1$, i.e.,

$$\int_a^b |x - t|^{(n-1)} |S(t, x)| dt = 2 \left[(x-a)^{n+1} + \left(\frac{a+b}{2} - x \right)^{n+1} \right] B(n, 2),$$

where $B(\cdot, \cdot)$ is the Euler beta function. To argue the sharpness, we consider first when $1 < p \leq \infty$, so that the equality in (2.1) holds when

$$f^{(n)}(t) = |x - t|^{(n-1)q-1} |S(t, x)|^{q-1} \operatorname{sgn} \left\{ (x-t)^{n-1} S(t, x) \right\},$$

thus the inequality (2.18) holds for $1 < p \leq \infty$. In case that $p = 1$, setting

$$g(t, x) = (x-t)^{n-1} S(t, x) \quad \forall x \in \left[a, \frac{a+b}{2} \right],$$

let t_0 be the point that gives the supremum. If $t_0 = \frac{x+(n-1)a}{n}$, we take

$$f_\varepsilon^{(n)}(t) = \begin{cases} \varepsilon^{-1}, & t \in (t_0 - \varepsilon, t_0) \\ 0, & \text{otherwise} \end{cases}.$$

Since

$$\begin{aligned} \left| \int_a^b g(t, x) f_\varepsilon^{(n)}(t) dt \right| &= \frac{1}{\varepsilon} \left| \int_{t_0-\varepsilon}^{t_0} g(t, x) dt \right| \leq \frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} |g(t, x)| dt \\ &\leq \sup_{t_0-\varepsilon \leq t \leq t_0} |g(t, x)| \cdot \frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} dt \\ &= |g(t_0, x)|, \end{aligned}$$

also, we have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} |g(t, x)| dt = |g(t_0, x)| = C(n, 1, x)$$

proving that $C(n, 1, x)$ is the best possible. \square

Corollary 1. *Under the assumptions of Theorem 4.*

(1) *If k is even and $f^{(k-1)}(a) = f^{(k-1)}(b) = 0$, for all $k = 1, \dots, n-1$. Then,*

$$(2.20) \quad \left| \frac{1}{n} \left(\frac{f(x) + f(a+b-x)}{2} \right) - \frac{1}{b-a} \int_a^b f(y) dy \right| \leq C(n, p, x) \|f^{(n)}\|_p.$$

(2) *If $f^{(k-1)}(a) = f^{(k-1)}(\frac{a+b}{2}) = f^{(k-1)}(b) = 0$, for all $k = 1, \dots, n-1$. Then the inequality (2.20) holds.*

Remark 1. *In Theorem 6, if one assumes that $f^{(n)}$ is n -convex, r -convex, quasi-convex, s -convex, P -convex, or Q -convex; we may obtain other new bounds involving convexity.*

3. BOUNDS VIA CHEBYSHEV-GRÜSS TYPE INEQUALITIES

The celebrated Čebyšev functional

$$(3.1) \quad \mathcal{C}(h_1, h_2) = \frac{1}{d-c} \int_c^d h_1(t) h_2(t) dt - \frac{1}{d-c} \int_c^d h_1(t) dt \cdot \frac{1}{d-c} \int_c^d h_2(t) dt.$$

has multiple applications in several subfields including Numerical integrations, Probability Theory & Statistics, Functional Analysis, Operator Theory and others. For more detailed history see [44].

The most famous bounds of the Čebyšev functional are incorporated in the following theorem:

Theorem 7. *Let $f, g : [c, d] \rightarrow \mathbb{R}$ be two absolutely continuous functions, then*

$$(3.2) \quad |\mathcal{C}(h_1, h_2)| \leq \begin{cases} \frac{(d-c)^2}{12} \|h'_1\|_\infty \|h'_2\|_\infty, & \text{if } h'_1, h'_2 \in L_\infty([c, d]), \quad \text{proved in [19]} \\ \frac{1}{4} (M_1 - m_1)(M_2 - m_2), & \text{if } m_1 \leq h_1 \leq M_1, \quad m_2 \leq h_2 \leq M_2, \quad \text{proved in [35]} \\ \frac{(d-c)}{\pi^2} \|h'_1\|_2 \|h'_2\|_2, & \text{if } h'_1, h'_2 \in L_2([c, d]), \quad \text{proved in [41]} \\ \frac{1}{8} (d-c) (M - m) \|h'_2\|_\infty, & \text{if } m \leq h_1 \leq M, \quad h'_2 \in L_\infty([c, d]), \quad \text{proved in [45]} \end{cases}$$

The constants $\frac{1}{12}$, $\frac{1}{4}$, $\frac{1}{\pi^2}$ and $\frac{1}{8}$ are the best possible.

In this section, we highlight the role of Čebyšev functional in integral approximations by using the Čebyšev-Grüss type inequalities (3.2).

Setting $h_1(t) = \frac{1}{n!} f^{(n)}(t)$ and $h_2(t) = (x-t)^{n-1} k(t, x)$, we have

$$\begin{aligned} \mathcal{C}(h_1, h_2) &= \frac{1}{n!(b-a)} \int_a^b (x-t)^{n-1} k(t, x) f^{(n)}(t) dt \\ &\quad - \frac{1}{n!} \cdot \frac{1}{b-a} \int_a^b (x-t)^{n-1} k(t, x) dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t) dt \\ &= \frac{1}{n!(b-a)} \int_a^b (x-t)^{n-1} k(t, x) f^{(n)}(t) dt \\ &\quad - \frac{2}{n!(b-a)} \left[(x-a)^{n+1} + \left(\frac{a+b}{2} - x \right)^{n+1} \right] B(n, 2) \\ &\quad \times \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \end{aligned}$$

which means

$$\begin{aligned} \mathcal{C}(h_1, h_2) &= \frac{1}{n} \left(\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} G_k(x) \right) - \frac{1}{b-a} \int_a^b f(y) dy \\ &\quad - \frac{2}{(n+1)!n(b-a)} \left[(x-a)^{n+1} + \left(\frac{a+b}{2} - x \right)^{n+1} \right] \\ &\quad \times \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \\ &:= \mathcal{P}(f; x, n). \end{aligned}$$

Theorem 8. Let I be a real interval, $a, b \in I^\circ$ ($a < b$). Let $f : I \rightarrow \mathbb{R}$ be $(n+1)$ -times differentiable on I° such that $f^{(n+1)}$ is absolutely continuous on I° with $(\cdot - t)^{n-1} k(t, \cdot) f^{(n)}(t)$ is integrable. Then, for all $n \geq 2$ we have

$$(3.3) \quad |\mathcal{P}(f; x, n)| \leq \begin{cases} (b-a)^2 \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{12n \cdot (n!)^2} \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^{n-1} \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]) \\ \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{4n \cdot (n!)^2} (2^{-n-2} - 2^{-2n-2}) (b-a)^{n-2} \cdot (M-m), & \text{if } m \leq f^{(n)} \leq M, \\ \frac{b-a}{(n!)^2 \pi^2} \sqrt{A(n)(x-a)^{2n-1} + B(n) \left(\frac{a+b}{2} - x \right)^{2n-1}} \cdot \|f^{(n+1)}\|_2, & \text{if } f^{(n+1)} \in L_2([a, b]), \\ (b-a) \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{8n \cdot (n!)^2} \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^{n-1} \cdot (M-m), & \text{if } m \leq f^{(n)} \leq M, \\ \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{8n \cdot (n!)^2} (2^{-n-2} - 2^{-2n-2}) (b-a)^n \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]), \end{cases}$$

holds for all $x \in [a, \frac{a+b}{2}]$, where

$$A(n) = \frac{2(n-1)^2}{(2n-1)(2n-2)(2n-3)}$$

and

$$B(n) = \frac{2^{2n-3} (2n-1)(2n-2) + 4n(2n-1) + 2n^2}{(2n-1)(2n-2)(2n-3)}$$

$\forall n \geq 2$.

Proof. • If $f^{(n+1)} \in L^\infty([a, b])$: Applying the first inequality in (3.2), it is not difficult to observe that $\sup_{a \leq t \leq b} \{|h'_1(t)|\} = \frac{1}{n!} \|f^{(n+1)}\|_\infty$ and

$$\sup_{a \leq t \leq b} \{|h'_2(t)|\} = \left(\frac{n-2}{n}\right)^{n-2} \frac{n^2 - 2n + 2}{n} \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^{n-1}, \quad \forall n \geq 2.$$

So that

$$|\mathcal{P}(f; x, n)| \leq (b-a)^2 \left(\frac{n-2}{n}\right)^{n-2} \frac{n^2 - 2n + 2}{12n \cdot n!} \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^{n-1} \cdot \frac{1}{n!} \|f^{(n+1)}\|_\infty.$$

• If $m \leq f^{(n)}(t) \leq M$, for some $m, M > 0$: Applying the second inequality in (3.2), we get

$$|\mathcal{P}(f; x, n)| \leq \frac{n^2 - 2n + 2}{4n \cdot n!} \left(\frac{n-2}{n}\right)^{n-2} (2^{-n-2} - 2^{-2n-2}) (b-a)^{n-2} \cdot \frac{1}{n!} (M-m).$$

• If $f^{(n+1)} \in L^2([a, b])$: Applying the third inequality in (3.2), we get

$$|\mathcal{P}(f; x, n)| \leq \frac{(b-a)}{n! \pi^2} \cdot \sqrt{A(n)(x-a)^{2n-1} + B(n) \left(\frac{a+b}{2} - x\right)^{2n-1}} \cdot \frac{1}{n!} \|f^{(n+1)}\|_2$$

$\forall n \geq 2$, where $A(n)$ and $B(n)$ are defined above.

• If $m \leq f^{(n)}(t) \leq M$, for some $m, M > 0$: Applying the forth inequality in (3.2), we get

$$|\mathcal{P}(f; x, n)| \leq (b-a) \left(\frac{n-2}{n}\right)^{n-2} \frac{n^2 - 2n + 2}{8n \cdot n!} \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^{n-1} \cdot \frac{1}{n!} (M-m).$$

By applying the forth inequality again the with dual assumptions, i.e., $f^{(n+1)} \in L^\infty([a, b])$, we have

$$|\mathcal{P}(f; x, n)| \leq \frac{n^2 - 2n + 2}{8n \cdot n!} \left(\frac{n-2}{n}\right)^{n-2} (2^{-n-2} - 2^{-2n-2}) (b-a)^n \cdot \frac{1}{n!} \|f^{(n+1)}\|_\infty.$$

Hence the proof is completely established. \square

Corollary 2. *Let assumptions of Theorem 8 hold. If moreover, $f^{(n-1)}(a) = f^{(n-1)}(b)$ ($n \geq 2$), then the inequality*

$$(3.4) \quad \left| \frac{1}{n} \left(\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} G_k \right) - \frac{1}{b-a} \int_a^b f(y) dy \right|$$

$$\leq \begin{cases} (b-a)^2 \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{12n \cdot (n!)^2} \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^{n-1} \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]) \\ \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{4n \cdot (n!)^2} (2^{-n-2} - 2^{-2n-2}) (b-a)^{n-2} \cdot (M-m), & \text{if } m \leq f^{(n)} \leq M, \\ \frac{b-a}{(n!)^2 \pi^2} \sqrt{A(n)(x-a)^{2n-1} + B(n)\left(\frac{a+b}{2} - x\right)^{2n-1}} \cdot \|f^{(n+1)}\|_2, & \text{if } f^{(n+1)} \in L_2([a, b]), \\ (b-a) \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{8n \cdot (n!)^2} \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^{n-1} \cdot (M-m), & \text{if } m \leq f^{(n)} \leq M, \\ \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{8n \cdot (n!)^2} (2^{-n-2} - 2^{-2n-2}) (b-a)^n \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]), \end{cases}$$

holds for all $x \in [a, \frac{a+b}{2}]$, where

$$A(n) = \frac{2(n-1)^2}{(2n-1)(2n-2)(2n-3)}$$

and

$$B(n) = \frac{2^{2n-3} (2n-1)(2n-2) + 4n(2n-1) + 2n^2}{(2n-1)(2n-2)(2n-3)}$$

$\forall n \geq 2$.

Remark 2. By setting $h_1(t) = \frac{1}{n!} f^{(n)}(t) k(t, x)$ and $h_2(t) = (x-t)^{n-1}$, we obtain that

$$(3.5) \quad \begin{aligned} \mathcal{C}(h_1, h_2) &= \frac{1}{n} \left(\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} G_k(x) \right) - \frac{1}{b-a} \int_a^b f(y) dy \\ &\quad - \frac{1}{n!} \cdot \frac{(x-a)^n - (x-b)^n}{n(b-a)} \cdot \frac{f^{(n)}(x) + f^{(n)}(a+b-x)}{2} \\ &:= \mathcal{Q}(f; x, n). \end{aligned}$$

Applying Theorem 7 Chebyshev type bounds for $\mathcal{Q}(f; x, n)$ can be proved. We shall omit the details.

4. GENERALIZATIONS OF THE RESULTS

In this section, generalization of the identity (2.1) via Harmonic sequence of polynomials through Fink's approach is considered. Generalizations of Guessab-Schmeisser formula integral formula (1.9) which is of Euler-Maclaurin type for symmetric values of real functions are established. Some norm inequalities of these generalized formulae with some special cases which are of great interests are also provided.

Theorem 9. *Let I be a real interval, $a, b \in I^\circ$ ($a < b$). Let P_k be a harmonic sequence of polynomials and let $f : I \rightarrow \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous on I for $n \geq 1$ with $P_{n-1}(t) S(t, \cdot) f^{(n)}(t)$ is integrable. Then we have the*

representation

$$(4.1) \quad \frac{1}{n} \left[\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} \left\{ T_k(x) + \widetilde{F}_k(a, b) \right\} \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ = \frac{(-1)^{n-1}}{(b-a)n} \int_a^b P_{n-1}(t) S(t, x) f^{(n)}(t) dt,$$

for all $x \in [a, \frac{a+b}{2}]$, where

$$(4.2) \quad T_k(x) = \frac{(-1)^k}{2} \left\{ P_k(x) f^{(k)}(x) + P_k(a+b-x) f^{(k)}(a+b-x) \right\}$$

$\widetilde{F}_k(a, b)$ is given in (1.10) and $S(t, x)$ as given in Theorem 4.

Proof. Fix $x \in [a, b]$. In the representation (1.9), replace b by $\frac{a+b}{2}$ we get

$$(4.3) \quad \frac{1}{n} \left[f(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) f^{(k)}(x) + 2 \sum_{k=1}^{n-1} \widetilde{F}_k\left(a, \frac{a+b}{2}\right) \right] - \frac{2}{b-a} \int_a^{\frac{a+b}{2}} f(y) dy \\ = \frac{2(-1)^{n-1}}{(b-a)n} \int_a^{\frac{a+b}{2}} \left(\int_y^x P_{n-1}(t) f^{(n)}(t) dt \right) dy,$$

where F_k is given in (1.10). As a second step, in the same formula (1.9) we replace every x by $a+b-x$ and a by $\frac{a+b}{2}$ for all $x \in [\frac{a+b}{2}, b]$, then f has the representation

$$(4.4) \quad \frac{1}{n} \left[f(a+b-x) + \sum_{k=1}^{n-1} (-1)^k P_k(a+b-x) f^{(k)}(a+b-x) \right. \\ \left. + 2 \sum_{k=1}^{n-1} \widetilde{F}_k\left(\frac{a+b}{2}, b\right) \right] - \frac{2}{b-a} \int_{\frac{a+b}{2}}^b f(y) dy \\ = \frac{2(-1)^{n-1}}{(b-a)n} \int_{\frac{a+b}{2}}^b \left(\int_y^{a+b-x} P_{n-1}(t) f^{(n)}(t) dt \right) dy,$$

Multiplying (4.3) and (4.4) by $\frac{1}{2}$ and then adding the corresponding equations, we get

$$(4.5) \quad \frac{1}{n} \left[\frac{f(x) + f(a+b-x)}{2} \right. \\ \left. + \frac{1}{2} \sum_{k=1}^{n-1} (-1)^k \left\{ P_k(x) f^{(k)}(x) + P_k(a+b-x) f^{(k)}(a+b-x) \right\} \right. \\ \left. + \sum_{k=1}^{n-1} \left\{ \widetilde{F}_k\left(a, \frac{a+b}{2}\right) + \widetilde{F}_k\left(\frac{a+b}{2}, b\right) \right\} \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ = \frac{(-1)^{n-1}}{(b-a)n} \left[\int_a^{\frac{a+b}{2}} \left(\int_y^x P_{n-1}(t) f^{(n)}(t) dt \right) dy + \int_{\frac{a+b}{2}}^b \left(\int_y^{a+b-x} P_{n-1}(t) f^{(n)}(t) dt \right) dy \right].$$

But since

$$\begin{aligned} & \widetilde{F}_k \left(a, \frac{a+b}{2} \right) + \widetilde{F}_k \left(\frac{a+b}{2}, b \right) \\ &= \frac{(-1)^k (n-k)}{b-a} \left[P_k(a) f^{(k-1)}(a) - P_k \left(\frac{a+b}{2} \right) f^{(k-1)} \left(\frac{a+b}{2} \right) \right] \\ & \quad + \left[P_k \left(\frac{a+b}{2} \right) f^{(k-1)} \left(\frac{a+b}{2} \right) - P_k(b) f^{(k-1)}(b) \right] \\ &= \widetilde{F}_k(a, b), \end{aligned}$$

then (4.5) becomes

$$\begin{aligned} (4.6) \quad & \frac{1}{n} \left[\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} \left\{ T_k(x) + \widetilde{F}_k(a, b) \right\} \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ &= \frac{(-1)^{n-1}}{(b-a)n} \left[\int_a^{\frac{a+b}{2}} \left(\int_y^x P_{n-1}(t) f^{(n)}(t) dt \right) dy + \int_{\frac{a+b}{2}}^b \left(\int_y^{a+b-x} P_{n-1}(t) f^{(n)}(t) dt \right) dy \right]. \end{aligned}$$

Also, the right hand-side can be simplified as shown in (2.14)–(2.16), i.e., we have

$$\begin{aligned} & \int_a^{\frac{a+b}{2}} \left(\int_y^x P_{n-1}(t) f^{(n)}(t) dt \right) dy + \int_{\frac{a+b}{2}}^b \left(\int_y^{a+b-x} P_{n-1}(t) f^{(n)}(t) dt \right) dy \\ &= \int_a^b P_{n-1}(t) S(t, x) f^{(n)}(t) dt, \end{aligned}$$

where

$$S(t, x) = \begin{cases} t-a, & t \in [a, x] \\ t-\frac{a+b}{2}, & t \in (x, a+b-x) \\ t-b, & t \in [a+b-x, b] \end{cases}.$$

for all $x \in [a, \frac{a+b}{2}]$, which gives the desired representation in (4.1). \square

Corollary 3. *Under the assumptions of Theorem 9, we have*

$$\begin{aligned} (4.7) \quad & \frac{1}{n} \left[\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} \{W(x, y) + F_k(y)\} \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ &= \frac{1}{(b-a)n!} \int_a^b (y-t)^{n-1} S(t, x) f^{(n)}(t) dt \end{aligned}$$

where

$$W(x, y) = \frac{(-1)^k}{2} \left\{ (x-y)^k f^{(k)}(x) + (a+b-x-y)^k f^{(k)}(a+b-x) \right\}$$

for all $x \in [a, \frac{a+b}{2}]$ and all $y \in [a, b]$, where $S(t, x)$ is given in Theorem 4 and $F_k(y)$ is given in (1.4)

Proof. In (4.1), choose $P_k(t) = \frac{(t-y)^k}{k!}$, we get the desired representation (4.7). \square

A Guessab–Schmeisser like expansion (see Theorem 3) may be deduced as follows:

Corollary 4. *Under the assumptions of Theorem 9. Additionally if $P_k(t) = (-1)^k P_k(a+b-t)$, $\forall t \in [a, b]$, then*

$$(4.8) \quad \frac{1}{n} \left[\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} \left\{ \widetilde{T}_k(x) + \widetilde{F}_k(a, b) \right\} \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ = \frac{(-1)^{n-1}}{(b-a)n} \int_a^b P_{n-1}(t) S(t, x) f^{(n)}(t) dt$$

where

$$\widetilde{T}_k(x) = \frac{(-1)^k}{2} P_k(x) \left[f^{(k)}(x) + (-1)^k f^{(k)}(a+b-x) \right],$$

for all $x \in [a, \frac{a+b}{2}]$.

Proof. Since $P_k(t) = (-1)^k P_k(a+b-t)$, $\forall t \in [a, b]$, substituting in (4.1) we get the required result. \square

It is convenient to remark here, from (4.8) we can deduce (4.1) by substituting $P_k(t) = \frac{(t-x)^k}{k!}$ in (4.8), so that we get

$$(4.9) \quad \frac{1}{n} \left[\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} \widetilde{F}_k(a, b) \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ = \frac{1}{n!(b-a)} \int_a^b (x-t)^{n-1} S(t, x) f^{(n)}(t) dt.$$

Clearly, the desired deduction is finished once we observe that $\widetilde{F}_k(a, b) = G_k(x)$. Since $P_k(b) = (-1)^k P_k(a)$, then

$$\begin{aligned} \widetilde{F}_k(a, b) &= \frac{(-1)^k (n-k)}{b-a} P_k(a) \left[f^{(k-1)}(a) - (-1)^k f^{(k-1)}(b) \right] \\ &= \frac{(-1)^k (n-k)}{b-a} \frac{(a-x)^k}{k!} \left[f^{(k-1)}(a) - (-1)^k f^{(k-1)}(b) \right] \\ &= \frac{(n-k)}{b-a} \frac{(x-a)^k}{k!} \left[f^{(k-1)}(a) + (-1)^{k+1} f^{(k-1)}(b) \right]. \end{aligned}$$

Also, we note that

$$\begin{aligned} P_k\left(\frac{a+b}{2}\right) &= \frac{\left(\frac{a+b}{2} - x\right)^k}{k!} = (-1)^k \frac{\left(x - \frac{a+b}{2}\right)^k}{k!} \\ &= (-1)^k P_k\left(\frac{a+b}{2}\right) = (-1)^k P_k\left(a+b - \frac{a+b}{2}\right), \end{aligned}$$

this gives that

$$0 = P_k\left(\frac{a+b}{2}\right) - (-1)^k P_k\left(\frac{a+b}{2}\right) = \left(1 + (-1)^{k+1}\right) P_k\left(\frac{a+b}{2}\right).$$

By our choice of P_k ; we have $P_k\left(\frac{a+b}{2}\right) = \frac{\left(\frac{a+b}{2}-x\right)^k}{k!}$, for all $x \in [a, \frac{a+b}{2}]$, therefore we can write

$$\begin{aligned}\widetilde{F}_k(a, b) + 0 &= \widetilde{F}_k(a, b) + \left(1 + (-1)^{k+1}\right) P_k\left(\frac{a+b}{2}\right) \\ &= \frac{(n-k)}{(b-a)k!} \left[(x-a)^k \left(f^{(k-1)}(a) + (-1)^{k+1} f^{(k-1)}(b) \right) \right. \\ &\quad \left. + \left(1 + (-1)^{k+1}\right) \left(\frac{a+b}{2} - x\right)^k \right] \\ &= G_k(x), \text{ which is given in (2.2).}\end{aligned}$$

Hence, the representation (4.8) reduces to (4.1).

Theorem 10. *Under the assumptions of Theorem 9, we have*

$$(4.10) \quad \left| \frac{1}{n} \left[\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} \left\{ T_k(x) + \widetilde{F}_k(a, b) \right\} \right] - \frac{1}{b-a} \int_a^b f(y) dy \right| \leq N(f; x, a, b) \cdot \left\| f^{(n)} \right\|_p,$$

$\forall p \in [1, \infty]$ and all $x \in [a, \frac{a+b}{2}]$, where

(4.11)

$$N(f; x, a, b) := \frac{1}{n(b-a)} \begin{cases} \sup_{a \leq t \leq b} \{ |P_{n-1}(t)| |S(t, x)| \}, & p = 1 \\ \left(\int_a^b |P_{n-1}(t)|^q |S(t, x)|^q dt \right)^{1/q}, & 1 < p < \infty \\ \int_a^b |P_{n-1}(t)| |S(t, x)| dt, & p = \infty \end{cases},$$

where $T_k(x)$ is given in (4.2) and $\widetilde{F}_k(a, b)$ is given in (1.10).

Proof. Utilizing the triangle integral inequality on the identity (4.1) and employing some known norm inequalities we get

$$\begin{aligned}& \left| \frac{1}{n} \left[\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} \left\{ T_k(x) + \widetilde{F}_k(a, b) \right\} \right] - \frac{1}{b-a} \int_a^b f(y) dy \right| \\ & \leq \frac{1}{n(b-a)} \int_a^b |P_{n-1}(t)| |S(t, x)| \left| f^{(n)}(t) \right| dt \\ & \leq \frac{1}{n(b-a)} \begin{cases} \left\| f^{(n)} \right\|_1 \sup_{a \leq t \leq b} \{ |P_{n-1}(t)| |S(t, x)| \}, & p = 1 \\ \left\| f^{(n)} \right\|_p \left(\int_a^b |P_{n-1}(t)|^q |S(t, x)|^q dt \right)^{1/q}, & 1 < p < \infty \\ \left\| f^{(n)} \right\|_\infty \int_a^b |P_{n-1}(t)| |S(t, x)| dt, & p = \infty \end{cases} \\ & = N(f; x, a, b) \left\| f^{(n)} \right\|_p, \quad \forall p, 1 \leq p \leq \infty\end{aligned}$$

where $N(f; x, a, b)$ is defined in (4.11), and this completes the proof. \square

Corollary 5. *Under the assumptions of Theorem 10, we have*

$$(4.12) \quad \left| \frac{1}{n} \left[\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} \left\{ T_k(x) + \widetilde{F}_k(a, b) \right\} \right] - \frac{1}{b-a} \int_a^b f(y) dy \right| \\ \leq \frac{1}{n} \left[\frac{1}{4} + \frac{|x - \frac{3a+b}{4}|}{b-a} \right] \cdot \|P_{n-1}\|_q \cdot \|f^{(n)}\|_p,$$

$\forall p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and all $x \in [a, \frac{a+b}{2}]$, where

$$\|P_{n-1}\|_q = \begin{cases} \sup_{a \leq t \leq b} \{|P_{n-1}(t)|\}, & q = \infty \\ \left(\int_a^b |P_{n-1}(t)|^q dt \right)^{1/q}, & 1 < q < \infty \\ \int_a^b |P_{n-1}(t)| dt, & q = 1 \end{cases}$$

Proof. In (4.10), it is easy to verify that

$$N(f; x, a, b) \leq \frac{1}{n(b-a)} \sup_{a \leq t \leq b} \{|S(t, x)|\} \cdot \|P_{n-1}\|_q \\ = \frac{1}{n} \left[\frac{1}{4} + \frac{|x - \frac{3a+b}{4}|}{b-a} \right] \cdot \|P_{n-1}\|_q,$$

$\forall q \in [1, \infty]$ and all $x \in [a, \frac{a+b}{2}]$. \square

Remark 3. In Theorem 10, if one assumes that $f^{(n)}$ is convex, r -convex, quasi-convex, s -convex, P -convex, or Q -convex; we may obtain other new bounds involving convexity.

Remark 4. Bounds for the generalized formula (4.1) via Chebyshev-Grüss type inequalities can be done by setting $h_1(t) = \frac{(-1)^{n-1}}{n} f^{(n)}(t)$ and $h_2(t) = P_{n-1}(t) S(t, x)$, therefore we have

$$\mathcal{C}(h_1, h_2) \\ = \frac{(-1)^{n-1}}{n(b-a)} \int_a^b P_{n-1}(t) S(t, x) f^{(n)}(t) dt \\ - \frac{1}{b-a} \int_a^b P_{n-1}(t) S(t, x) dt \times \frac{(-1)^{n-1}}{n(b-a)} \int_a^b f^{(n)}(t) dt \\ = \frac{1}{n(b-a)} \int_a^b P_{n-1}(t) S(t, x) f^{(n)}(t) dt \\ - \frac{1}{b-a} \int_a^b P'_n(t) S(t, x) dt \times \frac{(-1)^{n-1}}{n} \cdot \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}$$

$$\begin{aligned}
&= \frac{(-1)^{n-1}}{n(b-a)} \int_a^b P_{n-1}(t) S(t, x) f^{(n)}(t) dt \\
&\quad - \left[\frac{P_n(x) + P_n(a+b-x)}{2} - \frac{P_{n+1}(b) - P_{n+1}(a)}{b-a} \right] \\
&\quad \times \frac{(-1)^{n-1}}{n} \cdot \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \\
&= \mathcal{L}(f, P_n, x).
\end{aligned}$$

We left the representations to the reader.

Theorem 11. Let I be a real interval, $a, b \in I^\circ$ ($a < b$). Let $f : I \rightarrow \mathbb{R}$ be $(n+1)$ -times differentiable on I° such that $f^{(n+1)}$ is absolutely continuous on I° with $(\cdot - t)^{n-1} k(t, \cdot) f^{(n)}(t)$ is integrable. Then, for all $n \geq 2$ we have

$$(4.13) \quad |\mathcal{L}(f, P_n, x)| \leq \begin{cases} \frac{(b-a)^2}{12n} \|P_{n-1} + P_{n-2}S(\cdot, x)\|_\infty \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]) \\ \frac{1}{4n} (M_1 - m_1)(M_2 - m_2), & \text{if } m_1 \leq f^{(n)} \leq M_1, \\ \frac{b-a}{\pi^2 n} D(n, x) \cdot \|f^{(n+1)}\|_2, & \text{if } f^{(n+1)} \in L_2([a, b]), \\ \frac{b-a}{8n} \|P_{n-1} + P_{n-2}S(\cdot, x)\|_\infty \cdot (M_1 - m_1), & \text{if } m_1 \leq f^{(n)} \leq M_1, \\ \frac{b-a}{8n} (M_2 - m_2) \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]), \end{cases}$$

holds for all $x \in [a, \frac{a+b}{2}]$, where

$$M_2 := \max_{a \leq t \leq b} \{P_{n-1}(t) S(t, x)\}, \quad m_2 := \min_{a \leq t \leq b} \{P_{n-1}(t) S(t, x)\}$$

and

$$D(n, x) = \left(\int_a^b |P_{n-1}(t) + P_{n-2}(t) S(t, x)|^2 dt \right)^{1/2} \quad \forall n \geq 2.$$

Proof. The proof of the result follows directly by applying Theorem 3.2 to the functions $h_1(t) = \frac{(-1)^{n-1}}{n} f^{(n)}(t)$ and $h_2(t) = P_{n-1}(t) S(t, x)$ as shown previously in Remark 4 and the rest of the proof done using Theorem 7. \square

Corollary 6. *Let assumptions of Theorem 11 hold. If moreover, $f^{(n-1)}(a) = f^{(n-1)}(b)$ ($n \geq 2$), then the inequality*

$$(4.14) \quad \left| \frac{1}{n} \left[\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} \left\{ T_k(x) + \widetilde{F}_k(a, b) \right\} \right] - \frac{1}{b-a} \int_a^b f(y) dy \right|$$

$$\leq \begin{cases} \frac{(b-a)^2}{12n} \|P_{n-1} + P_{n-2}S(\cdot, x)\|_\infty \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]) \\ \frac{1}{4n} (M_1 - m_1)(M_2 - m_2), & \text{if } m_1 \leq f^{(n)} \leq M_1, \\ \frac{b-a}{\pi^2 n} D(n, x) \cdot \|f^{(n+1)}\|_2, & \text{if } f^{(n+1)} \in L_2([a, b]), \\ \frac{b-a}{8n} \|P_{n-1} + P_{n-2}S(\cdot, x)\|_\infty \cdot (M_1 - m_1), & \text{if } m_1 \leq f^{(n)} \leq M_1, \\ \frac{b-a}{8n} (M_2 - m_2) \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]), \end{cases}$$

holds for all $x \in [a, \frac{a+b}{2}]$.

5. QUADRATURE RULES AND ERROR BOUNDS

5.1. Representations of Quadratures. In viewing (2.1), the integral $\int_a^b f(y) dy$ can be expressed by the general quadrature rule:

$$(5.1) \quad \int_a^b f(y) dy = \mathcal{Q}_n(f, x) + \mathcal{E}_n(f, x)$$

where

$$(5.2) \quad \mathcal{Q}_n(f, x) := \frac{b-a}{n} \left(\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} G_k(x) \right),$$

and

$$(5.3) \quad \mathcal{E}_n(f, x) := -\frac{1}{n!} \int_a^b (x-t)^{n-1} k(t, x) f^{(n)}(t) dt.$$

for all $a \leq x \leq \frac{a+b}{2}$.

In particular cases, we have:

- If $x = a$, then

$$(5.4) \quad \int_a^b f(y) dy = \mathcal{Q}_n(f, a) + \mathcal{E}_n(f, a)$$

such that

$$\mathcal{Q}_n(f, a) := \frac{b-a}{n} \left(\frac{f(a) + f(b)}{2} + \sum_{k=1}^{n-1} G_k^a \right),$$

and

$$\mathcal{E}_n(f, a) := -\frac{1}{n!} \int_a^b (a-t)^{n-1} k(t, a) f^{(n)}(t) dt,$$

where,

$$G_k^a = \frac{(n-k)}{k!} \cdot (1 + (-1)^{k+1}) \left(\frac{b-a}{2} \right)^k f^{(k-1)} \left(\frac{a+b}{2} \right),$$

and $k(t, a) = t - \frac{a+b}{2}$, for all $t \in (a, b)$.

- If $x = \frac{3a+b}{4}$, then

$$(5.5) \quad \int_a^b f(y) dy = \mathcal{Q}_n \left(f, \frac{3a+b}{4} \right) + \mathcal{E}_n \left(f, \frac{3a+b}{4} \right)$$

such that

$$\mathcal{Q}_n \left(f, \frac{3a+b}{4} \right) := \frac{b-a}{n} \left(\frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} + \sum_{k=1}^{n-1} G_k^{\frac{3a+b}{4}} \right),$$

and

$$\mathcal{E}_n \left(f, \frac{3a+b}{4} \right) := -\frac{1}{n!} \int_a^b \left(\frac{3a+b}{4} - t \right)^{n-1} k \left(t, \frac{3a+b}{4} \right) f^{(n)}(t) dt,$$

where,

$$G_k^{\frac{3a+b}{4}} = \frac{(n-k)}{k!} \cdot \left(\frac{b-a}{4} \right)^k \left\{ \left[f^{(k-1)}(a) + (-1)^{k+1} f^{(k-1)}(b) \right] + \left(1 + (-1)^{k+1} \right) f^{(k-1)} \left(\frac{a+b}{2} \right) \right\},$$

and

$$k \left(t, \frac{3a+b}{4} \right) = \begin{cases} t-a, & t \in \left[a, \frac{3a+b}{4} \right] \\ t - \frac{a+b}{2}, & t \in \left[\frac{3a+b}{4}, \frac{a+3b}{4} \right] \\ t-b, & t \in \left[\frac{a+3b}{4}, b \right] \end{cases}.$$

- If $x = \frac{a+b}{2}$, then

$$(5.6) \quad \int_a^b f(y) dy = \mathcal{Q}_n \left(f, \frac{a+b}{2} \right) + \mathcal{E}_n \left(f, \frac{a+b}{2} \right),$$

such that

$$\mathcal{Q}_n \left(f, \frac{a+b}{2} \right) := \frac{b-a}{n} \left(f \left(\frac{a+b}{2} \right) + \sum_{k=1}^{n-1} G_k^{\frac{a+b}{2}} \right),$$

and

$$\mathcal{E}_n \left(f, \frac{a+b}{2} \right) := -\frac{1}{n!} \int_a^b \left(\frac{a+b}{2} - t \right)^{n-1} k \left(t, \frac{a+b}{2} \right) f^{(n)}(t) dt,$$

where,

$$G_k^{\frac{a+b}{2}} = \frac{(n-k)}{k!} \cdot \left(\frac{b-a}{2} \right)^k \left[f^{(k-1)}(a) + (-1)^{k+1} f^{(k-1)}(b) \right],$$

and

$$k \left(t, \frac{a+b}{2} \right) = \begin{cases} t-a, & t \in \left[a, \frac{a+b}{2} \right] \\ t-b, & t \in \left[\frac{a+b}{2}, b \right] \end{cases}.$$

A general quadrature rule via harmonic sequence of polynomials can be considered as follows:

$$(5.7) \quad \int_a^b f(y) dy = \mathcal{Q}_n(f, P_n, x) + \mathcal{E}_n(f, P_n, x), \quad \forall x \in \left[a, \frac{a+b}{2} \right]$$

where $\mathcal{Q}_n(f, P_n, x)$ is the quadrature formula given by

$$\mathcal{Q}_n(f, P_n, x) := \frac{b-a}{n} \left[\frac{f(x) + f(a+b-x)}{2} + \sum_{k=1}^{n-1} \left\{ T_k(x) + \widetilde{F}_k(a, b) \right\} \right],$$

with error term

$$\mathcal{E}_n(f, P_n, x) := -\frac{(-1)^{n-1}}{n} \int_a^b P_{n-1}(t) S(t, x) f^{(n)}(t) dt,$$

such that

$$T_k(x) = \frac{(-1)^k}{2} \left\{ P_k(x) f^{(k)}(x) + P_k(a+b-x) f^{(k)}(a+b-x) \right\}$$

and

$$\widetilde{F}_k(a, b) = \frac{(-1)^k (n-k)}{b-a} \left[P_k(a) f^{(k-1)}(a) - P_k(b) f^{(k-1)}(b) \right].$$

Remark 5. Identities (5.1) and (5.7) can be considered as Euler–Maclaurin type formulae for symmetric values.

Remark 6. As we mentioned at the end of introduction section, many authors used some key or general expansion formulas such as Euler–Maclaurin type formulae and Bernoulli polynomials (cf. [24]) to construct some quadrature rules of Newton–Cotes and Gauss types as done in Franjić works [30]–[34]. Our expansions, the identities (5.1) and (5.7) can be considered as general key formulae instead of those used in [30]–[34] to construct several quadrature formulas for an arbitrary n -th differentiable real function. The same remark holds for the formulae (1.5), (1.9) and (1.10).

5.2. Errors bounds via Chebyshev–Grüss type inequalities. In what follows, error bounds for the quadrature rules obtained in Section 5 are proved. The proof of these bounds can be deduced from Corollary 2 and Corollary 6.

Proposition 1. Let I be a real interval, $a, b \in I^\circ$ ($a < b$). Let $f : I \rightarrow \mathbb{R}$ be such that f is n -times differentiable function such that $f^{(n+1)}$ ($n \geq 2$) is absolutely continuous on $[a, b]$. If $f^{(n-1)}(a) = f^{(n-1)}(b)$, then for all $n \geq 2$ we have

$$(5.8) \quad |\mathcal{E}_n(f, a)| \leq \begin{cases} \frac{1}{2^{n-1}} \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{12n \cdot (n!)^2} (b-a)^{n+2} \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]) \\ \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{4n \cdot (n!)^2} (2^{-n-2} - 2^{-2n-2}) (b-a)^{n-1} \cdot (M-m), & \text{if } m \leq f^{(n)} \leq M, \\ \frac{(b-a)^{n+\frac{3}{2}}}{2^{n-\frac{1}{2}} \cdot (n!)^2 \cdot \pi^2} B^{\frac{1}{2}}(n) \cdot \|f^{(n+1)}\|_2, & \text{if } f^{(n+1)} \in L_2([a, b]), \\ \frac{1}{2^{n-1}} \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{8n \cdot (n!)^2} (b-a)^{n+1} \cdot (M-m), & \text{if } m \leq f^{(n)} \leq M, \\ \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{8n \cdot (n!)^2} (2^{-n-2} - 2^{-2n-2}) (b-a)^{n+1} \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]), \end{cases}$$

$$\begin{aligned}
(5.9) \quad & \left| \mathcal{E}_n \left(f, \frac{3a+b}{4} \right) \right| \\
& \leq \begin{cases} \frac{1}{4^{n-1}} \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{12n \cdot (n!)^2} (b-a)^{n+2} \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]) \\ \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{4n \cdot (n!)^2} (2^{-n-2} - 2^{-2n-2}) (b-a)^{n-1} \cdot (M-m), & \text{if } m \leq f^{(n)} \leq M, \\ \frac{(b-a)^{n+\frac{3}{2}}}{4^{n-\frac{1}{2}} \cdot (n!)^2 \cdot \pi^2} \sqrt{A(n) + B(n)} \cdot \|f^{(n+1)}\|_2, & \text{if } f^{(n+1)} \in L_2([a, b]), \\ \frac{(b-a)^{n+1}}{4^{n-1}} \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{8n \cdot (n!)^2} \cdot (M-m), & \text{if } m \leq f^{(n)} \leq M, \\ \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{8n \cdot (n!)^2} (2^{-n-2} - 2^{-2n-2}) (b-a)^{n+1} \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]), \end{cases}
\end{aligned}$$

$$\begin{aligned}
(5.10) \quad & \left| \mathcal{E}_n \left(f, \frac{a+b}{2} \right) \right| \\
& \leq \begin{cases} \frac{1}{2^{n-1}} \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{12n \cdot (n!)^2} (b-a)^{n+2} \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]) \\ \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{4n \cdot (n!)^2} (2^{-n-2} - 2^{-2n-2}) (b-a)^{n-1} \cdot (M-m), & \text{if } m \leq f^{(n)} \leq M, \\ \frac{(b-a)^{n+\frac{3}{2}}}{2^{n-\frac{1}{2}} \cdot (n!)^2 \cdot \pi^2} A^{\frac{1}{2}}(n) \cdot \|f^{(n+1)}\|_2, & \text{if } f^{(n+1)} \in L_2([a, b]), \\ \frac{1}{2^{n-1}} \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{8n \cdot (n!)^2} (b-a)^{n+1} \cdot (M-m), & \text{if } m \leq f^{(n)} \leq M, \\ \left(\frac{n-2}{n} \right)^{n-2} \frac{n^2-2n+2}{8n \cdot (n!)^2} (2^{-n-2} - 2^{-2n-2}) (b-a)^{n+1} \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]), \end{cases},
\end{aligned}$$

where

$$A(n) = \frac{2(n-1)^2}{(2n-1)(2n-2)(2n-3)}$$

and

$$B(n) = \frac{2^{2n-3} (2n-1)(2n-2) + 4n(2n-1) + 2n^2}{(2n-1)(2n-2)(2n-3)}, \quad \forall n \geq 2.$$

And finally

$$\begin{aligned}
(5.11) \quad & |\mathcal{E}_n(f, P_n, x)| \\
& \leq \begin{cases} \frac{(b-a)^3}{12n} \|P_{n-1} + P_{n-2}S(\cdot, x)\|_\infty \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]) \\ \frac{b-a}{4n} (M_1 - m_1)(M_2 - m_2), & \text{if } m_1 \leq f^{(n)} \leq M_1, \\ \frac{(b-a)^2}{\pi^2 n} D(n, x) \cdot \|f^{(n+1)}\|_2, & \text{if } f^{(n+1)} \in L_2([a, b]), \\ \frac{(b-a)^2}{8n} \|P_{n-1} + P_{n-2}S(\cdot, x)\|_\infty \cdot (M_1 - m_1), & \text{if } m_1 \leq f^{(n)} \leq M_1, \\ \frac{(b-a)^2}{8n} (M_2 - m_2) \cdot \|f^{(n+1)}\|_\infty, & \text{if } f^{(n+1)} \in L_\infty([a, b]), \end{cases}
\end{aligned}$$

where $D(n, x)$ and m_2, M_2 are defined in Theorem 11.

Remark 7. Other error bounds can be stated using (2.19), (4.11) and (4.12).

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